Modeling a metal–vapor buffer-gas hollow cathode discharge

A. J. Lichtenberg and M. A. Lieberman

Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California 94720

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The operation of a metal–vapor buffer-gas hollow cathode discharge is re-examined using modeling techniques that have been developed for electronegative plasma discharges. It is shown that a previously developed global model can be extended to give spatially resolved densities as well as more accurate values for the plasma parameters. A numerical calculation for a neon–copper (Ne–Cu) discharge is used to illustrate the results of the modeling technique. The density profiles of the neutral and ion components, and their variation with the buffer gas pressure, are similar to those found by numerical solution of the complete equations. © 2000 American Institute of Physics.

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I. INTRODUCTION

Hollow cathode glow discharges have been used extensively as a source of metal–atom and metal–ion laser lines.1–5 Both analytic6,7 and numerical8 techniques have been used for determining the properties of the discharges and their scaling with pressure and discharge current. The analytic technique that has been used is a global model in which the profiles of the various species, and consequently the scaling of plasma quantities with parameters. Numerical solutions, although giving reasonably accurate component profiles,5 do not directly determine the scaling of plasma quantities with parameters.

In a series of articles we have obtained approximate solutions for electronegative plasmas, consisting of three spatially varying charge species: positive ions, negative ions, and electrons.6–8 Although quite different from the discharges to be investigated here, which have four spatially varying species, two positive ion species, electrons, and neutral metal atoms, the models we have developed have versions that are applicable to these discharges. Certain approximations, following from the models, such as the uniformity of that part of the electron distribution producing the ionization, allow analytic solutions to be obtained.

The basic configuration of the hollow cathode is a cylinder of length L and diameter 2R with an internal cylindrical cathode. One end of the cylinder is an anode, while the other end can be either a cathode or another anode, depending on the application, but, providing L ≫ 2R, the operation is much the same. This follows because the plasma operates much like an anode, with the cathode sheath confined to a narrow annular layer between the cathode cylinder and the slightly smaller anode cylinder. The analysis, however, is simpler in rectangular coordinates in which 2R is the distance between parallel plates, which for the purposes of a calculation can also be 2πR wide. Rectangular coordinates have been used in the global modeling2 and also in our spatially varying models.5–8

The physical picture is that if the fast electrons generated by secondary emission (caused by ions striking the cathodes) are in the pressure range of multiple bounces between sheaths, they form a fairly uniform distribution, which produces uniform ionization of the background buffer gas. For the application of metal–atom and metal–ion lasers, buffer gas and metal ions striking the cathode also produce sputtered metal atoms which charge exchange with the buffer gas ions, to produce the metal ions. Charge exchange between the thermal buffer gas and thermal buffer gas ions is very rapid but without consequence in the bulk plasma. However, the very short mean free path for this process dominates the buffer gas ions in the sheath, reducing their energy at the cathode surface such that a smaller number of metal ions can produce most of the sputtering. Typical discharge parameters at high current in a neon–copper (Ne–Cu) ion laser are buffer gas density n_b ~ 10^{13} \text{cm}^{-3} (Ne), metal atom density n_m ~ 10^{14} \text{cm}^{-3} (Cu), n_b+ ~ 10^{14} \text{cm}^{-3}, and n_m+ on the order of n_m, with all quantities dependent on the current density (see below). The coefficients of secondary electron emission at the cathode, \gamma_{b+} and \gamma_{m+}, are typically below 0.05, as is the sputtering coefficient \xi_{b+}, while the sputtering coefficient of the metal ions generally has \xi_{m+} > 1.

In the following we do not attempt a detailed comparison of theory with experiment. Our purpose is to expose the techniques and to examine the differences between a spatially varying model and a global model. Consequently we use nominal values of the various parameters, partly taken from Ref. 2, which highlight the comparison with the model developed there.

Results from models, such as those presented here, cannot be expected to yield very close quantitative agreement with experiments, because of various factors omitted from the analysis. For example, the complete description of the electrons requires kinetic analysis.9,10 Axial variations may also be significant. Recent analysis of plasmas with both buffer gas and metal ions in other geometries have tended...
toward increased complexity, to more closely capture these effects. As we shall see in Sec. IV, point matching of plasma parameters along with the scaling provided by analytic formulas can be used to obtain quantitative comparisons.

II. BASIC EQUATIONS

Taking plane parallel geometry, for simplicity, the basic diffusion equations in the bulk plasma, for the densities of buffer gas ions \( n_{b^+} \), metal atoms \( n_m \), and metal ions \( n_{m^+} \), are

\[
-D_{b^+} \frac{d^2 n_{b^+}}{dx^2} = S_{b^+} - K n_{b^+} n_m,
\]

\[
-D_m \frac{d^2 n_m}{dx^2} = -K n_{b^+} n_m,
\]

\[
-D_{m^+} \frac{d^2 n_{m^+}}{dx^2} = K n_{b^+} n_m,
\]

where \( S_{b^+} \) is the source function for ionization of the buffer gas by electrons, and \( K \) is the charge exchange coefficient between buffer ions and metal atoms. This somewhat simplified set assumes that all of the metal ion flux is due to metal atoms sputtered from the cathode, and the reverse process of charge exchange between metal ions and buffer gas atoms can be neglected.

Following Warner et al. if we make a further simplifying assumption that the pressure range is sufficiently low that electrons leaving the cathodes bounce between the sheaths while losing their energy through ionizations, then the source can be approximated as a uniform quantity between the sheaths

\[
S_{b^+} = G \Gamma_e / R,
\]

where \( \Gamma_e \) is the electron flow entering the plasma from either sheath, \( R \) the half distance between sheaths, and \( G \) the number of electron–ion pairs generated per cathode electron by ionization of the buffer gas. The boundary conditions at \( x = R \) can then be written

\[
G \Gamma_e - \Gamma_{b^+} = K \int_0^R n_{b^+} n_m dx,
\]

\[
\Gamma_e - \Gamma_m = K \int_0^R n_{b^+} n_m dx,
\]

and

\[
\Gamma_{m^+} = K \int_0^R n_{b^+} n_m dx,
\]

where \( \Gamma_{b^+} \), \( \Gamma_m \), and \( \Gamma_{m^+} \) are outward flows, and \( \Gamma_e \) is the flow of entering sputtered atoms. The integrals on the right hand sides can be eliminated to give the two continuity equations

\[
\Gamma_e = \Gamma_{m^+} + \Gamma_m + \Gamma_{b^+}
\]

and

\[
\Gamma_e = \Gamma_{b^+} + \Gamma_{m^+}.
\]

Equation (8) has the obvious physical meaning that all the sputtered metal atoms entering the plasma must leave as either metal atoms or ions. Similarly, Eq. (5) equates the number of created positive ions to the number leaving, neglecting recombination.

The above plasma relations are closed by the cathode relations for secondary electron emission

\[
\gamma_e = \gamma_b + \Gamma_b + \gamma_m + \Gamma_m \quad \text{(10)}
\]

and sputtering

\[
\gamma_s = \xi_b + \Gamma_b + \xi_m + \Gamma_m \quad \text{(11)}
\]

where other less important effects, such as electron emission due to photons, are neglected. The controlling variable is the current density

\[
J e = \Gamma_{b^+} + \Gamma_{m^+} + \Gamma_e \quad \text{(12)}
\]

with all fluxes taken positively. In the usual operating regimes for these discharges the cathode fall is relatively constant, and therefore so are the secondary emission and sputtering coefficients. In this case, with these coefficients assumed known, the above set of equations can be solved, numerically, as done by Van Veldhuizen and de Hoog. Alternatively, various approximate solutions can be constructed, as developed in Sec. III.

III. APPROXIMATE SOLUTIONS

A. Global model

To solve the equations of Sec. II, Warner et al. used a global approximation that the fluxes are related to the average densities by diffusive lifetimes

\[
\Gamma_{b^+} = R \bar{n}_{b^+} / \tau_{b^+}, \quad \Gamma_m = R \bar{n}_m / \tau_m \quad \text{(13)}
\]

with the lifetimes taken from assumed sinusoidal profiles

\[
\tau_{b^+} = \frac{1}{D_{b^+}} \left( \frac{2R}{\pi} \right)^2, \quad \tau_m = \frac{1}{D_m} \left( \frac{2R}{\pi} \right)^2 \quad \text{(14)}
\]

From Eq. (7) the metal ion flux is then given by

\[
\Gamma_{m^+} = R K \bar{n}_{b^+} \bar{n}_m \quad \text{(15)}
\]

and \( \bar{n}_{b^+} \) and \( \bar{n}_m \) can be eliminated using Eq. (13). Together with the basic boundary flux equations one can solve for the ratio

\[
\beta = \Gamma_{m^+} / \Gamma_{b^+} \quad \text{(16)}
\]

obtaining, implicitly,

\[
\frac{\beta(1 + \beta)}{\xi_{b^+} + \beta(\xi_{m^+} - 1)} \approx \frac{K}{R} \frac{\tau_{b^+}}{\tau_{m^+}} \frac{J}{e} \quad \text{(17)}
\]

from which all the fluxes and densities can be obtained.

A global model of this sort does not, of course, give any indication of the actual profiles. As discussed in our work on electronegative discharges, the magnitudes of the unknown quantities as well as their scalings can then be significantly in error. As seen in these articles, approximate profiles can be obtained analytically, leading to significant improvement on the magnitudes and scalings, and we follow the approaches developed there.
B. Low current regime

From Eq. (1) the solution for \( n_{b+} \) can be obtained by noting that if the metal ion density is low, as it is for small currents, then the volume loss is negligible. The lowest order solution (with the source only) is approximately

\[
n_{b+}(x) = n_{b+}(0) \left( 1 - \frac{x^2}{R^2} \right).
\]  

(18)

Here we have assumed a thin sheath and that the pressure is sufficiently high that the edge density is small at \( x = R \). This can be improved by use of a Bohm flux edge condition. By evaluating the flux at \( x = R \) we have approximately

\[
\Gamma_{b+} = 2D_{b+}n_{b+}(0)/R.
\]  

(19)

At low densities, again ignoring the recombination, we also have

\[
\Gamma_{b+} = G \Gamma_n R
\]  

(20)

but we do not need to explicitly use this expression. As the metal density increases the solution to Eq. (1) has a flattened central region and steeper sides but is not much changed until the recombination flux dominates the flux to the walls. We need to consider such a solution at high current densities. The solution for \( n_m \) is obtained by substituting for \( n_{b+}(x) \) from Eq. (18) into Eq. (2)

\[
D_m \frac{d^2 n_m}{dx^2} + Kn_{b+}(0) \left( 1 - \frac{x^2}{R^2} \right)n_m.
\]  

(21)

We need not obtain the exact solution to this equation, as the solution is relatively insensitive to the exact form of \( n_{b+} \). To lowest order we therefore approximate

\[
n_{b+}(x) \approx \tilde{n}_{b+} \approx \frac{1}{2}n_{b+}(0)
\]  

in Eq. (21) to obtain a solution of the form

\[
n_m = A \cosh \sqrt{\frac{K\tilde{n}_{b+}}{D_m}} x
\]  

subject to the B.C.

\[
D_m \left. \frac{dn_m}{dx} \right|_{x=R} = \Gamma_s - \Gamma_m.
\]  

(23)

To relate \( \Gamma_m \) to \( n_m \) we make the additional assumption, as used in the numerical solution of Eqs. (1)–(3) in Ref. 5, that \( \Gamma_m \) is the outgoing part of a near-isotropic thermal Maxwellian distribution, giving

\[
\Gamma_m = n_m(R) v_{TM}/4.
\]  

(24)

Other assumptions are also possible, such as taking the outgoing flux to be more Bohm-like. This would modify the magnitude of the metal density but leave the fluxes essentially unchanged. Substituting for \( n_m(R) \) from Eq. (24) into Eq. (23) and using Eq. (22) we solve for \( n_m(R) \), obtaining

\[
n_m(R) = \frac{\Gamma_s \coth \sqrt{\frac{K\tilde{n}_{b+}}{D_m}} R}{\sqrt{D_m K\tilde{n}_{b+} + \frac{v_{TM}}{4} \coth \sqrt{\frac{K\tilde{n}_{b+}}{D_m}} R}}
\]  

(25)

and

\[
n_m(x) = n_m(R) \frac{\cosh \sqrt{\frac{K\tilde{n}_{b+}}{D_m}} x}{\cosh \sqrt{\frac{K\tilde{n}_{b+}}{D_m}} R}.
\]  

(26)

The hyperbolic form for the metal ion distribution was found numerically and, as expected, the central depression deepens with decreasing \( D_m \) (increasing pressure) and increasing \( \tilde{n}_{b+} \).\( \dagger \) The solution in Eq. (26) is not accurate near the interface between the plasma and sheath, where \( n_m(x) \) must fall, but this should not affect the macroscopic quantities greatly.

Using Eq. (3), the spatial dependence \( n_{m+}(x) \) can be found from a double integration

\[
n_{m+}(x) = -\frac{K}{D_m} \int \int n_{b+}(x)n_m(x)d^2x + Ax + B.
\]  

(27)

An approximate solution is obtained by again making the assumption \( n_{b+}(x) = \tilde{n}_{b+} \) and applying boundary conditions. As we see below this is not required to obtain self-consistent values for the various constants, which are determined by applying the cathode relations.

From Eqs. (12) and (8) we have

\[
\Gamma_{b+} = J/e - \Gamma_m - \Gamma_e = J/e - \Gamma_s + \Gamma_m,
\]  

(28)

where we have dropped the generally small term \( \Gamma_e \). Using Eq. (11) together with Eq. (28) we solve for \( \Gamma_{b+} \), obtaining

\[
\Gamma_{b+} = \frac{J}{e} \left[ 1 - \frac{1}{(\xi_{m+} - \xi_{b+})(1 - \tilde{n}_{m} v_{TM}/4)} \right]
\]  

(29)

where \( \tilde{n}_{m} = n_m(R)/\Gamma_s \) is obtained from Eq. (25). Substituting for \( \Gamma_{b+} \) in Eq. (28) we have

\[
\Gamma_e = \frac{J}{e} \left[ 1 - \frac{\xi_{b+}}{(\xi_{m+} - \xi_{b+})(1 - \tilde{n}_{m} v_{TM}/4)} \right],
\]  

(30)

and using Eqs. (24) and (25) together with Eq. (30)

\[
\Gamma_m = \frac{J}{e} \left[ 1 - \frac{\xi_{b+} n_{TM}/4}{(\xi_{m+} - \xi_{b+})(1 - \tilde{n}_{m} v_{TM}/4)} \right].
\]  

(31)

The additional relation to close the set is obtained from Eq. (19), which when substituted into Eq. (29) gives an implicit relation for \( \tilde{n}_{b+} \), and therefore for \( \Gamma_{b+} \), \( \Gamma_s \), and \( \Gamma_m \) from Eqs. (29) and (30). Note that these fluxes are all positive quantities. Although \( \xi_{m+} - \xi_{b+} \) is usually greater than unity, the second parenthesis, which is a normalized diffusion flux, is sufficiently small to keep the denominator non-negative. An approximate relation for \( \Gamma_{m+} \) is then found from Eqs. (27) and (26).
\[ \Gamma_{m+} = \left( K\bar{n}_{b+} D_m \right)^{1/2} n_m(R) \tanh \left( \frac{K\bar{n}_{b+} D_m}{D_m} R \right). \] (32)

The problem has been reduced to the calculation of known functions. We also have the general relation for \( \beta \), by dividing Eq. (32) by Eq. (19)

\[ \beta = \frac{\Gamma_{m+}}{\Gamma_{b+}} = \frac{\left( K\bar{n}_{b+} D_m \right)^{1/2} n_m(R) \tanh \left( \frac{K\bar{n}_{b+} D_m}{D_m} R \right)}{3D_b + \bar{n}_{b+} / R}. \] (33)

To get a physical picture of the scaling and to compare the results to the global model, we examine the low current limit where the metal components are perturbations. For \( (K\bar{n}_{b+} / D_m)^{1/2} R \ll 1 \) and \( (K\bar{n}_{b+} D_m)^{1/2} \ll E_{\text{int}} \) we find the simple relations \( n_m(R) \approx 41 \Gamma s / v_{\text{Tm}} \), from Eq. (25), and taking \( \Gamma s \approx \xi_{b+} / e \) from Eq. (30) Eq. (25) yields

\[ n_m(R) = \frac{4 \xi_{b+} J}{v_{\text{Tm}} e}. \] (34)

Substituting \( n_m(R) \) in Eq. (33), with \( \tanh \left( K\bar{n}_{b+} / D_m R \right) \) equal to its argument, yields

\[ \beta = \frac{4 K R^2 \xi_{b+} J}{3 D_b + \bar{n}_{b+} / v_{\text{Tm}} e}, \] (35)

which linearly increases with \( J \). In this approximation, from Eq. (29), we note that

\[ \Gamma_{b+} = J e, \] (36)

i.e., essentially all of the current is buffer ion current. From the global model, in the same approximation, we have from Eq. (17)

\[ \beta = \frac{4 K R^2 \xi_{b+} J}{3 D_b + \bar{n}_{b+} / v_{\text{Tm}} e}, \] (37)

which, like Eq. (35), increases linearly with \( J \) and has the same lowest order \( K \) and \( \xi_{b+} \) dependence. However, if we substitute the global values of \( \tau_{b+} \) and \( \tau_m \), from Eq. (14) Eq. (37) becomes

\[ \beta = \frac{2}{\pi} \frac{4 K R^2 \xi_{b+} J}{3 D_b + \bar{n}_{b+} / v_{\text{Tm}} e}. \] (38)

The \( \beta \) obtained from Eq. (38) is different from \( \beta \) in Eq. (35) by a factor \( R / \xi_m \) such that it is significantly larger at higher pressures when \( R / \xi_m \gg 1 \). The reason for this difference is immediately apparent by noting that for \( (K\bar{n}_{b+} / D_m)^{1/2} R \ll 1, \Gamma = n_{m} v_{\text{Tm}} / 4 \) such that \( \tau_{m} = 4 R / v_{\text{Tm}} \), as opposed to the value obtained in Eq. (14) by assuming a sinusoidal profile.

**C. High current regime**

At high currents, with high metal densities, the charge exchange which destroys buffer gas ions competes with their production by ionization, reducing the wall flux of buffer gas ions. For the somewhat different problem of an electronegative plasma, in which the destruction was due to recombination, we found an elliptic solution for the buffer gas ions with a flattened central region and steeper edges. We further found that it was reasonable to approximate the flattened distribution as

\[ n_{b+} = \begin{cases} n_{b+}(0) & -(R-d) < x < R-d \\ n_{b+}(0)f(x) & R-d < x < R \ (x > 0) \end{cases} \] (39)

where \( f(x) = 1 \) at \( x = R-d \) and \( f(x) = 0 \) at \( x = R \). As in that problem we can take \( f(x) \) to be parabolic, so that it matches the low density parabolic profile, as \( d \to R \)

\[ f(x) = \left( 1 - \frac{(x+d-R)^2}{d^2} \right). \] (40)

All of the approximations used in obtaining the currents and the metal profiles are the same as in the low current regime, except for the one change of Eq. (39) with Eq. (40) replacing Eq. (18).

One quantity that has been measured is an average value of \( n_{m}(x), \bar{n}_{m} \), which rises linearly with \( J \) at small \( J \), goes through a transition of more rapid increase with \( J \), and then returns to a linear proportionality at a higher value. Approximating the profiles as fixed, we substitute Eq. (30) in Eq. (25) and divide by Eq. (34) to obtain the offset (ratio) of \( n_{m}(R) \), normalized to its low current limit

\[ \rho = \frac{\bar{n}_{m} v_{\text{Tm}}/4}{1 - (\xi_{m+} - \xi_{b+}) (1 - \bar{n}_{m} v_{\text{Tm}}/4)}, \] (41)

which is sensitive to parameters, and can be large. From the global model it is quite straightforward to show that \( n_{b+}(0) \) reaches a constant, asymptotically, a result that is physically reasonable since the increasing \( n_{m} \) finally makes charge exchange balance the production of \( n_{b+} \). For the general form of \( \Gamma_{b+} \) in Eq. (29) this implies that

\[ 1 - \xi_{m+} (1 - \bar{n}_{m} v_{\text{Tm}}/4) \to 0 \] (42)

and therefore \( \bar{n}_{m} v_{\text{Tm}}/4 \to 1 - 1/\xi_{m+} \). Substituting these limiting values into Eq. (41) we obtain

\[ \rho \to \frac{\xi_{m+} - 1}{\xi_{b+}}, \] (43)

which is the asymptotic (normally large) offset between the high and low current linear increases in the metal density with increasing current. Note that the numerics are sensitive to the choice of \( \xi_{m+} \) and \( \xi_{b+} \), which are not precisely known but can be adjusted to give the experimental value of \( \rho \) as has been done in Ref. 2. We note that within our approximations, Eq. (43) is the same whether the parabolic or flat profile is used.

To make one consistent set of formulas that are valid both at low and high current densities we need only produce a continuous transition from low to high density. This is readily done by noting that the transition from a parabolic to a flat-topped distribution occurs when the recombination flux competes with the diffusion flux, as noted in Ref. 8. Substituting the parabolic form from Eq. (40) in the diffusion equation, the diffusion flux in Eq. (19) becomes

\[ \Gamma_{b+} (R) = 2 D_{b+} n_{b+}(0) / d. \] (44)

Using the approximation that \( n_{b+}(x) = \bar{n}_{b+} = \gamma_{b+}(0) \), substituting for \( n_{m}(x) \) from Eq. (26) and integrating over the recombination, using Eq. (7)
We can then determine $d$ from a reasonable ansatz

$$\Gamma_{b+}^{\text{(rec)}} = \frac{2}{3} n_{b+}(0) \frac{K D_m}{n_m} n_m(R) \tanh \sqrt{\frac{K n_{b+} R}{D_m}}. \quad (45)$$

We can then determine $d$ from a reasonable ansatz

$$\frac{R - d}{R} = \frac{\Gamma_{b+}^{\text{(rec)}}}{\Gamma_{b+}^{\text{(R)}} + \Gamma_{b+}^{\text{(rec)}}}, \quad (46)$$

yielding

$$d = \frac{-2 D_m + \sqrt{(2 D_m)^2 + 8 D_m \Gamma_{b+}^{\text{(rec)}} R n_{b+}(0)}}{2 \Gamma_{b+}^{\text{(rec)}} / n_{b+}(0)} \quad (47)$$

which can be used in Eq. (44) to replace Eq. (19). Since $d \rightarrow R$ for $\Gamma_{b+}^{\text{(rec)}} \ll \Gamma_{b+}^{\text{(R)}}$, Eq. (19) is the low recombination limit of Eq. (44), such that Eq. (44) can be used for all currents. The approximations allow straightforward numerical solutions, which we present in Sec. IV. The use of Eq. (45), and also the use of Eq. (19) at low currents, implies that the mean free path $\lambda_{b+} < R$. This is true for the radial distance and pressure used in the example in Sec. IV.

If $\xi_{m+} < 1$, the solutions at high current are quite different. The sputtering of metal atoms by metal ions can no longer sustain the current, so that the production of buffer gas ions must continue to increase with increasing current. This was already recognized from the global model.\(^2\) The result is seen quite clearly in Eq. (29), since the factor in the numerator never goes to zero for $\xi_m < 1$, then $\Gamma_{b+}$ and therefore $n_{b+}(0)$ continue to increase proportional to $J$, while from Eq. (25), at large $n_{b+}$, we find $n_m(R) \propto J^{1/2}$. Substitut-

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**FIG. 1.** Comparison of global (dashed curves) with space varying (solid curves) model, with a gas temperature of 300 °K. The buffer ion density (a), the metal atom density (b), and the ratio of metal ion to buffer ion densities (c) are plotted against the current density.

**FIG. 2.** The same as Fig. 1 except that the metal atom velocity and corresponding metal ion diffusion coefficient are increased to correspond to a gas temperature of 1000 °K rather than a temperature of 300 °K in Fig. 1.
ing these proportionalities into Eq. (33), we find that $\beta \to \text{const}$ at large currents. Although these scalings are of less practical importance than those for $\zeta_m > 1$, for completeness we shall numerically compare the results to those found from the global model in Sec. IV.

IV. NUMERICAL CALCULATIONS AND CONCLUDING REMARKS

We use some estimates developed by Warner et al.\textsuperscript{2} for their global model. We consider a Ne–Cu discharge with a half width of $R=0.1\text{ cm}$ and a Ne pressure of $p=7.5\text{ Torr}$. The thermal velocity of the Cu atoms is taken to be, at the gas temperature, $v_{\text{Tm}} = 3.2 \times 10^4 \text{ cm/s}$ for $T_g = 300\text{ K}$, or $v_{\text{Tm}} = 5.9 \times 10^4 \text{ cm/s}$ for $T_g = 10^3\text{ K}$ measured in Ref. 5. The values of $\zeta_{m+} = 10^{-2}$ and $\zeta_m = 1.5$, estimated from various experimental results, are somewhat different from the values used in Ref. 2, but give the same value of $\rho$. The value of $K = 3 \times 10^{-9}\text{ cm}^3\text{s}^{-1}$ is estimated from experiments. From the nominal time constants taken by fitting the global model to experimental data,\textsuperscript{2} the diffusion constants are estimated, from Eq. (14), to be $D_{m+} = 0.8 \times 10^2\text{ cm}^2\text{s}^{-1}$ and $D_m = 2.7 \times 10^2\text{ cm}^2\text{s}^{-1}$. These latter values are, of course, nominal, both because the time constants are obtained from curve fitting, and because they are not constants in the real problem, as described in Sec. III. The current range examined is in the range $J = 10^{-3} - 1\text{ A/cm}^2$. We note that $\lambda_{b+} = D_{b+} / v_{Tb+} \approx 2.3 \times 10^{-4}\text{ cm}$, which is quite small compared to $R$. 

![FIG. 3. Spatial profiles of the metal atom density: (a) relative profile on a log scale, (b) absolute profile on a log scale, (c) absolute profile on a linear scale. From top to bottom in (a) and from bottom to top in (b) and (c) $J=0.01, 0.02, 0.04, 0.08, 0.12\text{ A/cm}^2$.](image1)

![FIG. 4. The same as Fig. 1 except that the secondary emission ratio for metal atoms leaving the surface vs metal ions striking the surface is $\zeta = 0.5$, rather than $\zeta = 1.5$ in Fig. 1.](image2)
In Figs. 1 and 2 we compare the global model (dashed curves) to the space varying model (solid curves), for the two values of $v_{T_m} = 3.2 \times 10^4$ and $v_{T_m} = 5.9 \times 10^4$, respectively. For the latter velocity, $D_m$ is increased to $5 \times 10^2 \text{ cm}^2 \text{s}^{-1}$ for consistency. Significant differences are found both in the high current asymptotes and in the current at which the transitions occur. We see that the asymptotic value of $\bar{n}_{b+}$ is an order of magnitude larger from the spatially varying model than for the global model. This results from the decay of $n_m$ into the interior, resulting in significantly less charge exchange. The values if $\bar{n}_m$ at low values of $J$ are quite close to the value found from a global model, as they should be when the metal profile is broad. While $\rho$ is the same as obtained in the global model, by construction, the offset ratio of $\bar{n}_m$ between high and low $J$ is much smaller due to this effect. Similarly, the values of $\beta$ are lower with the spatially varying model. The differences are accentuated if the higher value of $v_{T_m}$ is used with the lower value of $D_m$ taken from Ref. 2. The value of $D_m$ is probably too small even for the lower value of $v_{T_m}$.

In Fig. 3 the metal profiles are shown for a few selected currents for the parameters of Fig. 1. A logarithmic ratio of $n_m(x)/n_m(R)$ is given for clarity in (a), and $n_m(x)$ is shown on a logarithmic scale in (b) and a linear scale in (c) to emphasize that the integral under the curves, which determines $\bar{n}_m$, can increase much more slowly than $n_m(R)$. The curves are for $J = 0.01$, 0.02, 0.04, 0.08, and 0.12 A/cm$^2$.

Numerically, the profiles found by van Veldhuizen and de Hoog\textsuperscript{5} over a more limited current range, have profiles similar to those in Fig. 3 at similar pressure and current. However, since the current range examined by them, from $J = 0.1$ A/cm$^2$ to $J = 1$ A/cm$^2$ are all in a relatively constant $n_{b+}(0)$ regime, relatively little change in profile was seen with changing current in their numerical and experimental results. It would be quite useful to examine these profiles, numerically, in a lower current range.

In Fig. 4 the comparison of Fig. 1 is repeated for $\xi_{m+}$ = 0.5. For this case $\bar{n}_{b+}$ does not saturate. The reason for this difference can be understood in that $\bar{n}_{b+}$ continually increases as the current increases, since the metal ion density cannot sustain the discharge by itself.

In conclusion, we have found that there are significant differences between a global model and a space-varying model for a metal–vapor hollow cathode discharge. Although the general character of the discharge, as the current is varied, is evident from the global model, specific quantities, such as the density of the buffer ions at saturation, may be considerably in error. Because the analytic form of the space-varying model presented here requires its own set of approximations, it would be interesting to compare these results to a complete numerical solution of the basic equations.

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